Quarks, Triality, and Unitary Symmetry Schemes*

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It is shown that integral eigenvalues of Q/e can be achieved for Gell-Mann's quarks (particles belonging to representations of SU_3 with nonzero triality type) by the introduction of a new additive quantum number for strongly interacting particles and a corresponding alteration of the Gell-Mann–Nishijima formula. A possible framework wherein such ideas can be formulated is provided by an extension of SU_3 symmetry to SU_4 or perhaps a higher unitary symmetry.

THE purpose of this paper is to give a brief discussion of three related subjects:

(a) The existence of 'quarks' in the Ne'eman,[1] Gell-Mann[2] theory of unitary symmetry.

(b) The concept of triality type for irreducible representations of SU_3.

(c) The existence of a higher unitary symmetry, or even of a hierarchy of such symmetries.

To give the motivation for our work we introduce, following Biedenharn and Fowler,[3] the concept of the triality type for SU_3. We use the Young diagram notation[4] $\{l\}=\{l_1,l_2\}$ for irreducible representations of SU_3, and define the triality type t_l of $\{l\}$ by

$$t_l=l_1+l_2(\text{modulo } 3). \tag{1}$$

We may conventionally take $+1$, 0, -1 as the allowed values of t_l. The importance of triality type stems from the fact that for all $\{l\}$ contained in the direct product $\{m\}\otimes\{n\}$, we have

$$t_l=t_m+t_n(\text{modulo } 3), \tag{2}$$

so that

$$T_l=\exp(2\pi i t_l/3) \tag{3}$$

is a conserved multiplicative quantum number. All the representations of SU_3 so far used in the unitary symmetry theory (the singlet $\{0,0\}$, the octet $\{2,1\}$, and the decuplet $\{3,0\}$) have $t=0$. Recently, however, Gell-Mann[5] has discussed the possible existence in nature of particles (quarks) belonging to the triplet representations $\{1,0\}$ and $\{1,1\}$ of SU_3. If one retains the usual association of isospin and hypercharge operators with generators of SU_3 and the Gell-Mann–Nishijima formula

$$Q/e=I_z+\tfrac{1}{2}Y, \tag{4}$$

it then follows that quarks have nonintegral eigenvalues of Q/e. Such a statement obtains, in fact, for any representation of SU_3 which, like $\{1,0\}$ and $\{1,1\}$, has $t\neq0$. Biedenharn and Fowler[3] sought to avoid this unattractive conclusion by replacing Eq. (4) by

$$Q/e=I_z+\tfrac{1}{2}Y+t/3. \tag{5}$$

This does lead to integral eigenvalues of Q/e for all t and reduces, as it should, to (4) for $t=0$. However, it implies that Q/e is conserved only modulo 3, which seems to the authors to make the definition (5) unacceptable.

In this paper, we suggest an alternative possibility which is not open to this latter objection. Our procedure will be seen to correlate the existence in nature of particles belonging to representations of SU_3 with $t\neq0$ with the possible existence of unitary symmetries higher than SU_3.

We begin our discussion by reviewing the course of events which led to the introduction of the SU_3-symmetry scheme. Early studies of low-energy nuclear physics and pion physics served to establish isospin conservation or SU_2 invariance, and subsequent analysis of high-energy events suggested the existence of a hypercharge conservation law. From the recognition of $SU_2\otimes U_1$, where U_1 is the hypercharge gauge group, as the symmetry of the strong interactions, it was not unnatural to suggest that the strong interactions possess at least approximate SU_3 invariance. If we define the duality d of an irreducible representation[6] of SU_2 by

$$d=2I(\text{modulo } 2), \tag{6}$$

then we can have $d=0$ and 1, corresponding respectively to isobosons and isofermions. Since all representations of SU_3 except the identity contain $d=0$ and $d=1$ representations of SU_2, it follows that, if only isobosons had been observed in nature, the possibility of an SU_3 symmetry scheme would have been excluded.

We now discuss the consequences of the possible discovery of a new additive quantum number in the study of events at higher energies than are at present accessible. Let us call it $Y^{(3)}$ and distinguish it from the known hypercharge by writing $Y=Y^{(2)}$. This would give

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[1] Y. Ne'eman, Nucl. Phys. **26**, 222 (1961).

[2] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962).

[3] L. C. Biedenharn and Earle C. Fowler, 1963 (to be published). See also G. E. Baird and L. C. Biedenharn, paper presented at the Conference on Symmetry Principles at High Energy, University of Miami, January 1964 (unpublished).

[4] The relationship of Young diagram and highest weight notations for irreducible representations of SU_n is explained in a paper currently being prepared by the authors. If $\{l_1,l_2\}$ and (λ_1,λ_2) refer to the same irreducible representation of SU_3 in the two notations, then the l_i and λ_i are related by $l_1=\lambda_1+\lambda_2$ and $l_2=\lambda_2$.

[5] M. Gell-Mann, Phys. Letters **8**, 214 (1964).

[6] Use of $\lambda=2I$ in place of I as a label for irreducible representations of SU_2 would bring the notation used for SU_2 into line with that used for other special unitary groups.

$SU_3 \otimes U_1^{(3)}$ as the underlying symmetry group of the strong interactions, $U_1^{(3)}$ being the gauge group generated by $Y^{(3)}$. Then it would surely be tempting to postulate full SU_4 invariance.[7] However, all but the one dimensional representation of SU_4 contain within them $t=\pm1$ in addition to $t=0$ representations of SU_3, so that the possibility of SU_4 invariance depends crucially upon the discovery of particles of nonzero triality. Further, if $t\neq0$ states do exist, they must have nonintegral eigenvalues of Q/e as long as definition (4) is retained. Of course, if we view the SU_3 symmetry scheme suggested here as an incomplete manifestation of an SU_4-symmetry scheme, we are led naturally to the following generalization of the Gell-Mann–Nishijima relationship (4):

$$Q/e=I_z+Y^{(2)}/2+Y^{(3)}/3. \tag{7}$$

We shall show that Eq. (7)

(a) reduces to Eq. (4) for observed particles,

(b) leads to integral eigenvalues of Q/e for states of all representations of $SU_3 \otimes U_1^{(3)}$ contained within representations of SU_4 with zero quadrality.[8]

Let us refer to the irreducible representations of SU_4 in the Young diagram notation as $\{m\}^{(3)}=\{m_1,m_2,m_3\}$ and to those of SU_3 as $\{l\}^{(2)}=l_1,l_2\}$. Then the quadrality k of $\{m\}^{(3)}$ is defined by

$$k=m_1+m_2+m_3 \text{ (modulo 4)}, \tag{8}$$

and

$$K=\exp\{2\pi ik/4\} \tag{9}$$

is a multiplicatively conserved quantum number. To establish our claims we shall use the following theorem.[9] The irreducible representation $\{m\}^{(3)}$ of SU_4 contains within it the representation

$$\{l_1-l_3,\, l_2-l_3\}\otimes Y^{(3)}, \tag{10}$$

where

$$Y^{(3)}=l_1+l_2+l_3-3(m_1+m_2+m_3)/4 \tag{11}$$

of $SU_3 \otimes U_1^{(3)}$ once and only once for each triplet of integers l_1, l_2, and l_3 allowed by the inequalities

$$m_1\geqslant l_1\geqslant m_2\geqslant l_2\geqslant m_3\geqslant l_3\geqslant 0. \tag{12}$$

Proof of (b) is almost immediate. From (1), (8), and (11) and the fact that the m_i and l_i are positive integers it follows that $Y^{(3)}/3$ and $t/3-k/4$, where t is the triality of $\{l_1-l_3,\, l_2-l_3\}$ and k is the qaudrality of $\{m\}^{(3)}$, differ only by an integer. In particular, when $k=0$, $Y^{(3)}/3$ and $t/3$ differ only by an integer. Hence, since Eq. (5) leads to integral Q/e eigenvalues, so also does Eq. (7). Equation (7), however, corresponds to Q being a strictly conserved additive quantum number.

We now turn to point (a). It is, of course, not possible at present to make any definite statements. But, if SU_4 symmetry exists, one would expect that the known SU_3 multiplets correspond to the representations $\{0,0\}\otimes0$, $\{2,1\}\otimes0$, and $\{3,0\}\otimes0$ of $SU_3\otimes U^{(3)}$ contained within low-dimensional $k=0$ representations of SU_4. Surveying the available representations of SU_4 of this type and using the above theorem to determine which representations of $SU_3\otimes U_1^{(3)}$ are contained in them, it is readily seen that suitable vacancies exist, so that (7) can be used in the SU_4 symmetry theory without being in contradiction with currently accepted quantum number assignments. More specific remarks cannot usefully be made until particles are discovered which might occupy the $t\neq0$ states that are necessarily found within an SU_4 symmetry scheme.

The point of view here under study can be pursued further. If no $t\neq0$ representations of SU_3 are ever discovered, then, of course, no SU_4 theory is tenable. If, however, they are found in sufficient number and with a $Y^{(3)}$ quantum number, one can pass to an SU_4-symmetry theory as described here using only $k=0$ representations. Then the same question arises: Do $k\neq0$ representations occur, and if they do, is their occurrence associated with yet another conserved quantum number $Y^{(4)}$? An analogous treatment of the question exists; one envisages the SU_4 theory as part of an SU_5 theory using only representations of zero quintality and redefines Q by the inclusion of a term $1/4\ Y^{(4)}$ on the right side of Eq. (7) in such a way that Q/e, for those $k=0$ representations of SU_4 used in the theory, is still given by Eq. (7) [i.e., $Y^{(4)}=0$ for them], and that Q/e has integral eigenvalues for $k\neq0$.[10] Again the same situation arises. Of course the chain of possibilities just mentioned awaits evidence at the level of Gell-Mann's suggestion that $t\neq0$ states are of physical consequence.

[7] P. Tarjanne and V. L. Teplitz, Phys. Rev. Letters **11**, 441 (1963), have discussed the possibility that the strong interactions possess SU_4 as an approximate symmetry group. See also a preprint by P. Roman, Boston University (unpublished).

[8] The generalization of triality for SU_3 to SU_n is discussed in the paper cited in footnote 4.

[9] This is a special case of a theorem stated and proved in the paper cited in footnote 4.

[10] It has been emphasized by Dr. Okubo in a private communication that if one considers the group U_3, it is possible to accommodate Gell-Mann's quarks without the undesirable feature of fractional charge. This does not nullify our conclusion concerning the necessity of nonzero triality multiplets in higher unitary symmetry schemes. One might further expect that any group which contains SU_3 as a subgroup (e.g., U_3) might be able to include quarks of integral charge. In view, however, of the dominant role played by the special unitary groups in particle physics and their natural suggestion of a possible hierarchy of unitary symmetries, we have seen fit to single out the SU_n groups in this work.